11. Gram-Schmidtova procedura

(11.01) Uvod u problem

Cilj: Iskoristiti datu bazu $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ i uz pomoć nje konstruisati ortonormiranu bazu $\mathcal{O} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ za $\mathcal{S}$.

Strategija: Postepeno konstruisati $\mathcal{O}$ tako da je $\mathcal{O}_k = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k\}$ ortonormirana baza za $\mathcal{S}_k = span\{x_1, x_2, ..., x_k\}$ za $k = 1, ..., n$. ◇

(11.02) Gram-Schmidtov proces ortogonalizacije

Ako je $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ baza za neki unitarni prostor $\mathcal{S}$, tada *Gram-Schmidtov niz* definisan sa

$$\boldsymbol{u}_1 = \frac{x_1}{\|\boldsymbol{x}_1\|} \quad \text{i} \quad \boldsymbol{u}_k = \frac{x_k - \sum_{i=1}^{k-1}\langle \boldsymbol{u}_i, \boldsymbol{x}_k\rangle \boldsymbol{u}_i}{\|x_k - \sum_{i=1}^{k-1}\langle \boldsymbol{u}_i, \boldsymbol{x}_k\rangle \boldsymbol{u}_i\|} \text{ za } k = 2, ..., n$$

je ortonormirana baza za $\mathcal{S}$. Kada je $\mathcal{S}$ n-dimenzionalni podprostor od $\mathbb{C}^m$, Gram-Schmidtov niz se može izraziti sa

$$\boldsymbol{u}_k = \frac{(I - U_k U_k^*)\boldsymbol{x}_k}{\|(I - U_k U_k^*)\boldsymbol{x}_k\|} \quad \text{za} \quad k = 1, 2, ..., n$$

gdje je $U_1 = \mathbf{0} \in \mathbb{C}^m$ i $U_k = (\boldsymbol{u}_1|\boldsymbol{u}_2|...|\boldsymbol{u}_{k-1})_{m\times k-1}$ za $k > 1$. ◇

(11.03) Klasični Gram-Schmidtov algoritam

Sljedeći algoritam je direktna ili "klasična" implementacija Gram-Schmidtove procedure. Oznaka $a \leftarrow b$ znači da "a definiši da bude (ili postaje) b."

Za $k = 1$:

$$\boldsymbol{u}_1 \leftarrow \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$$

Za $k > 1$:

$$\boldsymbol{u}_k \leftarrow \boldsymbol{x}_k - \sum_{i=1}^{k-1}(\boldsymbol{u}_i^*\boldsymbol{x}_k)\boldsymbol{u}_i$$

$$\boldsymbol{u}_k \leftarrow \frac{\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|}$$

◇

(11.04) QR faktorizacija

Svaka matrica $A_{m\times n}$ sa linearno nezavisnim kolonama se može jedinstveno faktorisati kao $A = QR$ gdje su kolone od $Q_{m\times n}$ ortonormirana baza za $im(A)$ a $R_{n\times n}$ je gornje trougaona matrica sa pozitivnim dijagonalnim vrijednostima.

• QR faktorizacija je potpun "opis" Gram-Schmidtove procedure zato što su kolone od $Q = (\boldsymbol{q}_1|\boldsymbol{q}_2|...|\boldsymbol{q}_n)$ dobijene kao rezultat primjene Gram-Schmidtovog procesa na kolone matrice $A = (\boldsymbol{a}_1|\boldsymbol{a}_2|...|\boldsymbol{a}_n)$ a matrica R je data sa

$$R = \begin{bmatrix} \nu_1 & \boldsymbol{q}_1^*\boldsymbol{a}_2 & \boldsymbol{q}_1^*\boldsymbol{a}_3 & ... & \boldsymbol{q}_1^*\boldsymbol{a}_n \\ 0 & \nu_2 & \boldsymbol{q}_2^*\boldsymbol{a}_3 & ... & \boldsymbol{q}_2^*\boldsymbol{a}_n \\ 0 & 0 & \nu_3 & ... & \boldsymbol{q}_3^*\boldsymbol{a}_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & ... & \nu_n \end{bmatrix}$$

gdje je $\nu_1 = \|\boldsymbol{a}_1\|$ i $\nu_k = \|\boldsymbol{a}_k - \sum_{i=1}^{k-1}\langle \boldsymbol{q}_i, \boldsymbol{a}_k\rangle \boldsymbol{q}_i\|$ za $k > 1$. ◇

(11.05) Linearni sistemi i QR faktorizacija

Ako je $rang(A_{m\times n}) = n$, i ako je $A = QR$ dobijena QR faktorizacija, tada rješenje nesingularnog trougaonog sistema

$$R\boldsymbol{x} = Q^\top \boldsymbol{b}$$

je ili riješenje problema najmanjih kvadrata sistema $A\boldsymbol{x} = \boldsymbol{b}$ ili riješenje istog sistema u zavisnosti da li je $A\boldsymbol{x} = \boldsymbol{b}$ saglasan sistem. $\diamond$

(11.06) Modifikovani Gram-Schmidtov algoritam

Za linearno nezavisan skup $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\} \subseteq \mathbb{C}^m$ Gram-Schmidtov niz dat u 11.02 se može napisati i na drugi način kao

$$\boldsymbol{u}_k = \frac{E_k ... E_2 E_1 \boldsymbol{x}_k}{\|E_k ... E_2 E_1 \boldsymbol{x}_k\|} \quad \text{gdje je } E_1 = I, \;\; E_i = I - \boldsymbol{u}_{i-1}\boldsymbol{u}_{i-1}^* \text{ za } \; i > 1,$$

i ovaj niz je generisan pomoću sljedećeg algoritma:

Za $k = 1$: $\boldsymbol{u}_1 \leftarrow \boldsymbol{x}_1/\|\boldsymbol{x}_1\|$ i $\boldsymbol{u}_j \leftarrow \boldsymbol{x}_j$ za $j = 2, 3, ..., n$.
Za $k > 1$: $\boldsymbol{u}_j \leftarrow E_k\boldsymbol{u}_j = \boldsymbol{u}_j - (\boldsymbol{u}_{k-1}^*\boldsymbol{u}_j)\boldsymbol{u}_{k-1}$ za $j = k, k+1, ..., n$.
$\boldsymbol{u}_k \leftarrow \boldsymbol{u}_k/\|\boldsymbol{u}_k\|$. $\diamond$

(11.07) Sažetak

• Kada Gram-Schmidtov proces (klasičan ili modifikovan) primjenimo na kolone matrice A koristeći tačnu aritmetiku, svaki put dobijamo ortonormiranu bazu za $im(A)$.

• Za računanje QR faktorizacije u aritmetici pokretnog zareza, modifikovani algoritam proizvodi rezultate koji su dovoljno dobri a često i bolji od klasičnog algoritma, ali modificirani algoritam nije bezuslovno stabilan - postoje situacije u kojima će proizvesti skup kolona koje nisu ni približno ortogonalne.

• Za rješenje problema najmanjeg kvadrata sa aritmetikom pokretnog zareza, modificirana procedura je numerički stabilan algoritam u smislu da metoda opisana u jednom od primjera vraća rezultat koji je tačno rješenje susjednog problema najmanjih kvadrata. Kakogod, Hauseholderova metoda (koju nismo radili ali možete tražiti papire da kopirate od predmetnog nastavnika ili predmetnog asistenta) je dovoljno stabilna a potrebno joj je dosta manje aritmetičkih operacija. $\diamond$

Koristeći klasičnu formulaciju Gram-Schmidt-ove procedure pronaći ortonormiranu bazu za prostor generisan pomoću sljedeća tri linearno nezavisna vektora

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Rj.

Klasični Gram-Schmidt-ov algoritam:

Za $k=1$

$$u_1 \leftarrow \frac{x_1}{\|x_1\|}$$

Za $k>1$

$$u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle x_k, u_i \rangle u_i$$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

gdje je tumačenje za $a \leftarrow b$: "a definiši da bude b" ili "a postaje b".

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \|x\|_1^2 = \langle x_1, x_1 \rangle = x_1^T x_1 = (1 \ 0 \ 0 \ -1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 2$$

$$\|x\|_1 = \sqrt{2}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$u_2' = x_2 - \underbrace{\langle x_2, u_1 \rangle}_{x_2^T u_1} u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} - (1 \ 2 \ 0 \ -1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \cdot 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \|u_2'\|^2 = 4$$

$$\|u_2'\| = 2$$

$$u_2 = \frac{1}{2}\begin{pmatrix}0\\2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$u_3' = x_3 - \langle x_3, u_1\rangle u_1 - \langle x_3, u_2\rangle u_2$$

$$\langle x_3, u_1\rangle = x_3^T u_1 = (3\ \ 1\ \ 1\ \ -1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot(3+0+0+1) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\langle x_3, u_2\rangle = x_3^T u_2 = (3\ \ 1\ \ 1\ \ -1)\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 1$$

$$u_3' = \begin{pmatrix}3\\1\\1\\-1\end{pmatrix} - 2\cancel{\sqrt{2}}\cdot\frac{1}{\cancel{\sqrt{2}}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} - \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\1\\1\\-1\end{pmatrix} + \begin{pmatrix}-2\\0\\0\\2\end{pmatrix} + \begin{pmatrix}0\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}$$

$$\|u_3'\|^2 = 3$$

$$\|u_3'\| = \sqrt{3} \qquad u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$$

Prema tome

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$$

je tražena ortonormirana baza.

grčko slovo

ν n_i

(#) Odrediti QR faktorizaciju matrice

$$A = \begin{pmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{pmatrix}.$$

Rj. Svaku matricu $A = (a_1 | a_2 | \dots | a_n)$ sa linearno nezavisnim kolonama $a_1, a_2, \dots, a_n$ možemo jedinstveno faktorisati kao

$$A = QR = (q_1 | q_2 | \dots | q_n) \begin{pmatrix} \nu_1 & \langle a_2, q_1 \rangle & \langle a_3, q_1 \rangle & \dots & \langle a_n, q_1 \rangle \\ 0 & \nu_2 & \langle a_3, q_2 \rangle & \dots & \langle a_n, q_2 \rangle \\ 0 & 0 & \nu_3 & \dots & \langle a_n, q_3 \rangle \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & \nu_n \end{pmatrix}$$

gdje je $q_1 = \dfrac{a_1}{\nu_1}$, $q_k = \dfrac{a_k - \sum_{i=1}^{k-1} \langle a_k, q_i \rangle q_i}{\nu_k}$, $\nu_1 = \|a_1\|$,

$$\nu_k = \| a_k - \sum_{i=1}^{k-1} \langle a_k, q_i \rangle q_i \|.$$

Prvo provjerimo da su kolone matrice A linearno nezavisne. Ovo je ekvivalentno sa provjerom da je $\det A \neq 0$. (objasniti zašto?)

$$|A| = \begin{vmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{vmatrix} = (-2) \begin{vmatrix} 0 & -20 & 7 \\ 3 & 27 & 2 \\ 4 & 11 & 1 \end{vmatrix} \underset{II_v - III_v \cdot 2}{\overset{I_v - III_v \cdot 7}{=}} (-2) \begin{vmatrix} -28 & -97 & 0 \\ -5 & 5 & 0 \\ 4 & 11 & 1 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} -28 & -97 \\ -5 & 5 \end{vmatrix} = (-2) \cdot 5 \begin{vmatrix} -28 & -97 \\ -1 & 1 \end{vmatrix} = (-10) \cdot (-28 - 97) \neq 0$$

Prema tome vektori $a_1 = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$, $a_2 = \begin{pmatrix} -20 \\ 27 \\ 11 \end{pmatrix}$ i $a_3 = \begin{pmatrix} -14 \\ -4 \\ -2 \end{pmatrix}$ su linearno nezavisni.

$$\|a_1\|^2 = (0\ 3\ 4) \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 25 \qquad q_1 = \frac{1}{5} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}, \quad \nu_1 = 5$$

$\|a_1\| = 5$

$$g_2 = \frac{a_2 - \langle a_2, g_1\rangle g_1}{\| a_2 - \langle a_2, g_1 \rangle g_1 \|}$$

$$\langle a_2, g_1 \rangle = (-20 \quad 27 \quad 11) \cdot \frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{5}(81+44) = \frac{1}{5}\cdot 125 = 25$$

$$a_2 - \langle a_2, g_1 \rangle g_1 = \begin{pmatrix} -20 \\ 27 \\ 11 \end{pmatrix} - 25\,\frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -20 \\ 27 \\ 11 \end{pmatrix} + \begin{pmatrix} 0 \\ -15 \\ -20 \end{pmatrix} = \begin{pmatrix} -20 \\ 12 \\ -9 \end{pmatrix}$$

$$\| a_2 - \langle a_2, g_1 \rangle g_1 \|^2 = 400 + 144 + 81 = 625$$

$$\| a_2 - \langle a_2, g_1 \rangle g_1 \| = 25$$

$$g_2 = \frac{1}{25}\begin{pmatrix} -20 \\ 12 \\ -9 \end{pmatrix}, \qquad \nu_2 = 25$$

$$g_3 = \frac{a_3 - \langle a_3, g_1 \rangle g_1 - \langle a_3, g_2 \rangle g_2}{\| a_3 - \langle a_3, g_1 \rangle g_1 - \langle a_3, g_2 \rangle g_2 \|}$$

$$\langle a_3, g_1 \rangle = (-14 \quad -4 \quad -2)\,\frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{5}(-12-8) = -4$$

$$\langle a_3, g_2 \rangle = (-14 \quad -4 \quad -2)\,\frac{1}{25}\begin{pmatrix} -20 \\ 12 \\ -9 \end{pmatrix} = \frac{1}{25}(280 - 48 + 18) = \frac{250}{25} = 10$$

$$a_3 - \langle a_3, g_1 \rangle g_1 - \langle a_3, g_2 \rangle g_2 = \begin{pmatrix} -14 \\ -4 \\ -2 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} - \frac{10}{25}\begin{pmatrix} -20 \\ 12 \\ -9 \end{pmatrix} =$$

$$= \frac{1}{25}\left(25\begin{pmatrix} -14 \\ -4 \\ -2 \end{pmatrix} + 20\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} - 10\begin{pmatrix} -20 \\ 12 \\ -9 \end{pmatrix} \right) = \frac{1}{25}\begin{pmatrix} -150 \\ -160 \\ 120 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -30 \\ -32 \\ 24 \end{pmatrix}$$

$$a_3 - \langle a_3, g_1 \rangle g_1 - \langle a_3, g_2 \rangle g_2 = \frac{2}{5}\begin{pmatrix} -15 \\ -16 \\ 12 \end{pmatrix}$$

$$\| a_2 - \langle a_3, g_1 \rangle g_1 - \langle a_2, g_2 \rangle g_2 \|^2 = \frac{4}{25}(-15 \quad -16 \quad 12)\begin{pmatrix} -15 \\ -16 \\ 12 \end{pmatrix} = \frac{4}{25}\cdot 625 = 4\cdot 25 = 100$$

$$\| a_3 - \langle a_3, g_1 \rangle g_1 - \langle a_3, g_2 \rangle g_2 \| = 10$$

$$g_3 = \frac{1}{25}\begin{pmatrix} -15 \\ -16 \\ 12 \end{pmatrix}, \quad \nu_3 = 10$$

$$A = \begin{pmatrix} 0 & -\frac{20}{25} & -15/25 \\ 3/5 & 12/25 & -16/25 \\ 4/5 & -9/25 & 12/25 \end{pmatrix}\begin{pmatrix} 5 & 25 & -4 \\ 0 & 25 & 10 \\ 0 & 0 & 10 \end{pmatrix}$$

tražena faktorizacija

(#) Koristeći aritmetiku sa 3-decimalna mjesta primjeniti modificirani Gram-Schmidt-ov algoritam na skup

$$x_1 = \begin{pmatrix} 1 \\ 10^{-3} \\ 10^{-3} \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}$$

Rj: Klasični Gram-Schmitov algoritam nije dobar u slučaju kada se pojavi broj sa decimalnim zarezom, tj. nije baš dobar numerički algoritam. U tom slučaju koristimo modificirani Gram-Schmidtov algoritam:

Za $k=1$: $u_1 \leftarrow \frac{x_1}{\|x_1\|}$

$u_j \leftarrow x_j$ za $j=2,3,\dots,n$

Za $k>1$: $u_j \leftarrow E_k u_j = u_j - \langle u_j, u_{k-1} \rangle u_{k-1}$ za $j=k,k+1,\dots,n$

$u_k \leftarrow \frac{u_k}{\|u_k\|}$

Drugačiji opis ovog algoritma je sljedeći

$$\{x_1, x_2, \dots, x_n\} \xrightarrow{\text{normalizirati 1-vi vektor}} \{u_1, x_2, \dots, x_n\}$$

$$\xrightarrow{\text{primjeniti } E_2} \{u_1, E_2 x_2, E_2 x_3, \dots, E_2 x_n\}$$

$$\xrightarrow{\text{normalizirati 2gi vektor}} \{u_1, u_2, E_2 x_3, \dots, E_2 x_n\}$$

$$\xrightarrow{\text{primjeniti } E_3} \{u_1, u_2, E_3 E_2 x_3, \dots, E_3 E_2 x_n\}$$

$$\xrightarrow{\text{normalizirati 3-ći vektor}} \{u_1, u_2, u_3, E_3 E_2 x_3, \dots, E_3 E_2 x_n\}$$

itd.

gdje je $E_1 = I$, $E_i = I - u_{i-1} u_{i-1}^*$ za $i>1$

$$x_1=\begin{pmatrix}1\\10^{-3}\\10^{-3}\end{pmatrix},\qquad u_1=\frac{E_1x_1}{\|E_1x_1\|},\quad \text{gdje je}\quad E_1=I$$

$$\|x_1\|^2=x_1^Tx_1=(1\ 10^{-3}\ 10^{-3})\begin{pmatrix}1\\10^{-3}\\10^{-3}\end{pmatrix}=1+10^{-6}+10^{-6}\approx 1$$

$$u_1=\begin{pmatrix}1\\10^{-3}\\10^{-3}\end{pmatrix}$$

$$x_2=\begin{pmatrix}1\\10^{-3}\\0\end{pmatrix},\qquad u_2=\frac{E_2E_1x_2}{\|E_2E_1x_2\|},\quad \text{gdje je } E_1=I,\ E_i=I-u_{i-1}u_{i-1}^T$$

za $i>1$

$$E_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1\\10^{-3}\\10^{-3}\end{bmatrix}\begin{bmatrix}1&10^{-3}&10^{-3}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&10^{-3}&10^{-3}\\10^{-3}&10^{-6}&10^{-6}\\10^{-3}&10^{-6}&10^{-6}\end{bmatrix}$$

$$E_2=\begin{bmatrix}0&-10^{-3}&-10^{-3}\\-10^{-3}&1-10^{-6}&-10^{-6}\\-10^{-3}&-10^{-6}&1-10^{-6}\end{bmatrix}\approx\begin{bmatrix}0&-10^{-3}&-10^{-3}\\-10^{-3}&1&0\\-10^{-3}&0&1\end{bmatrix}$$

$$E_2E_1x_2=\begin{bmatrix}0&-10^{-3}&-10^{-3}\\-10^{-3}&1&0\\-10^{-3}&0&1\end{bmatrix}\begin{bmatrix}1\\10^{-3}\\0\end{bmatrix}=\begin{bmatrix}-10^{-6}\\0\\-10^{-3}\end{bmatrix}\approx\begin{bmatrix}0\\0\\-10^{-3}\end{bmatrix}$$

$$\|E_2E_1x_2\|^2=(0\ 0\ -10^{-3})\begin{pmatrix}0\\0\\-10^{-3}\end{pmatrix}=10^{-6}$$

$$\|E_2E_1x_2\|=10^{-3}$$

$$u_2=\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$$u_3=\frac{1}{10^{-3}}\begin{bmatrix}0\\-10^{-3}\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Prema tome modificirana Gram-Schmitova procedura daje vektore $u_1=\begin{pmatrix}1\\10^{-3}\\10^{-3}\end{pmatrix}$, $u_2=\begin{pmatrix}0\\0\\-1\end{pmatrix}$ i $u_3=\begin{pmatrix}0\\-1\\0\end{pmatrix}$

$$x_3=\begin{pmatrix}1\\0\\10^{-3}\end{pmatrix},\quad u_3=\frac{E_3E_2E_1x_3}{\|E_3E_2E_1x_3\|},\quad \text{gdje je } E_1=I,\quad E_i=I-u_{i-1}u_{i-1}^T,\ i>1$$

$$E_3=I-u_2u_2^T=I-\begin{bmatrix}0\\0\\-1\end{bmatrix}\begin{bmatrix}0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

$$E_3E_2E_1x_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&-10^{-3}&-10^{-3}\\-10^{-3}&1&0\\-10^{-3}&0&1\end{bmatrix}\begin{bmatrix}1\\0\\10^{-3}\end{bmatrix}=\begin{bmatrix}0&-10^{-3}&-10^{-3}\\-10^{-3}&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\10^{-3}\end{bmatrix}=\begin{bmatrix}-10^{-6}\\-10^{-3}\\0\end{bmatrix}\approx\begin{bmatrix}0\\-10^{-3}\\0\end{bmatrix}$$

$$\|E_3E_2E_1x_3\|^2=10^{-6}\Rightarrow\|E_3E_2E_1x_3\|=10^{-3}$$

(#) Neka je $\mathcal{S} = \text{span}\left\{ x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}$.

a) Uz pomoć klasičnog Gram-Schmidtovog algoritma (sa tačnom aritmetikom) odrediti ortonormiranu bazu za $\mathcal{S}$.

b) Direktno provjeriti da Gram-Schmidtov niz, proizveden pod (a), je zaista ortonormirana baza za $\mathcal{S}$.

c) Ponoviti dio pod a) koristeći modifikovani Gram-Schmidtov algoritam, i uporediti rezultate.

Rj.

a) Ako je $\mathcal{B} = \{x_1, x_2, \dots, x_n\}$ baza za unitarni prostor $\mathcal{S}$, tada Gram-Schmidtov niz definisan sa

$$u_1 = \frac{x_1}{\|x_1\|} \quad i \quad u_k = \frac{x_k - \sum_{i=1}^{k-1} \langle x_k, u_i \rangle u_i}{\left\| x_k - \sum_{i=1}^{k-1} \langle x_k, u_i \rangle u_i \right\|} \quad \text{za } k = 2, 3, \dots, n$$

je ortonormirana baza za $\mathcal{S}$.

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \|x_1\|^2 = (1\ 1\ 1\ -1)\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 4 \qquad u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\|x_1\| = 2$$

$$x_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{x_2 - \langle x_2, u_1 \rangle u_1}{\| x_2 - \langle x_2, u_1 \rangle u_1 \|}$$

$$\langle x_2, u_1 \rangle = x_2^T u_1 = (2\ -1\ -1\ 1)\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}\cdot(-1) = -\frac{1}{2}$$

$$x_2 - \langle x_2, u_1 \rangle u_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{4}\left[\begin{pmatrix} 8 \\ -4 \\ -4 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix} 9 \\ -3 \\ -3 \\ 3 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\|x_2 - \langle x_2, u_1 \rangle u_1\|^2 = \frac{3}{4}(3\ -1\ -1\ 1)\cdot\frac{3}{4}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{9}{16}\cdot(9+1+1+1) = \frac{9}{\cancel{16}_{\cancel{8}\,4}}\cdot\cancel{12}^{\cancel{6}\,3} = \frac{9}{4}\cdot 3$$

$$\|x_2 - \langle x_2, u_1 \rangle u_1\| = \frac{3}{2}\sqrt{3}$$

$$u_2 = \frac{3}{4}\cdot\frac{2}{3\sqrt{3}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad u_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \frac{x_3 - \langle x_3, u_1 \rangle u_1 - \langle x_3, u_2 \rangle u_2}{\|x_3 - \langle x_3, u_1 \rangle u_1 - \langle x_3, u_2 \rangle u_2\|}$$

$$\langle x_3, u_1\rangle = (-1\ 2\ 2\ 1)\cdot\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} = \frac{1}{2}(-1+2+2-1) = \frac{1}{2}\cdot 2 = 1$$

$$\langle x_3, u_2\rangle = (-1\ 2\ 2\ 1)\cdot\frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{2\sqrt{3}}(-3-2-2+1) = \frac{1}{2\sqrt{3}}\cdot(-6) = -\frac{3}{\sqrt{3}}$$

$$x_3 - \langle x_3,u_1\rangle u_1 - \langle x_3,u_2\rangle u_2 = \begin{pmatrix}-1\\2\\2\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} + \frac{3}{\sqrt{3}}\cdot\frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} =$$

$$= \frac{1}{2}\left[\begin{pmatrix}-2\\4\\4\\2\end{pmatrix} + \begin{pmatrix}-1\\-1\\-1\\+1\end{pmatrix} + \begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}0\\2\\2\\4\end{pmatrix} = \begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

$$\| x_3 - \langle x_3,u_1\rangle u_1 - \langle x_3,u_2\rangle u_2 \|^2 = 1+1+4 = 6$$

$$u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

Prema tome ortonormirana baza za $\mathcal{S}$ je

$$u_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix},\quad u_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix},\quad u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

b) provjera

$$\langle u_1, u_1\rangle = \frac{1}{2}(1\ 1\ 1\ -1)\cdot\frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} = \frac{1}{4}\cdot 4 = 1$$

$$\langle u_1, u_2\rangle = \frac{1}{2}(1\ 1\ 1\ -1)\cdot\frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{4\sqrt{3}}\cdot 0 = 0$$

$$\langle u_1, u_3\rangle = \frac{1}{2}(1\ 1\ 1\ -1)\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix} = \frac{1}{2\sqrt{6}}\cdot 0 = 0$$

$$\langle u_3, u_3\rangle = \frac{1}{\sqrt{6}}(0\ 1\ 1\ 2)\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix} = \frac{1}{6}\cdot 6 = 1$$

$$\langle u_3, u_2\rangle = \frac{1}{\sqrt{6}}(0\ 1\ 1\ 2)\cdot\frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{2\sqrt{18}}(0-1-1+2) = 0$$

$$\langle u_2, u_2\rangle = \frac{1}{2\sqrt{3}}(3\ -1\ -1\ 1)\frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{12}\cdot 12 = 1$$

c) Modifikovan Gram-Schmidtov algoritam izgleda ovako:

$k=1$: $u_1 \leftarrow \frac{x_1}{\|x_1\|}$, $u_j \leftarrow x_j$ za $j = 2,3,\dots,n$

$k>1$: $u_j \leftarrow E_k u_j = u_j - (u_{k-1}^* u_j)\, u_{k-1}$ za $j = k, k+1, \dots, n$

$$u_k \leftarrow \frac{u_k}{\|u_k\|} \qquad\qquad E_i = I - u_{i-1}u_{i-1}^* \quad i>1$$

$$\mathcal{B} = \left\{ x_1 = \begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix}, x_2 = \begin{pmatrix} 2\\-1\\-1\\1 \end{pmatrix}, x_3 = \begin{pmatrix} -1\\2\\2\\1 \end{pmatrix} \right\}$$

$k=1$:

$$x_1 = \begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix}, \quad \|x_1\| = \sqrt{\langle x_1, x_1 \rangle} = \sqrt{4} = 2$$

$$u_1 \leftarrow \frac{x_1}{\|x_1\|} = \frac{1}{2}\begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix}$$

$$u_2 \leftarrow \begin{pmatrix} 2\\-1\\-1\\1 \end{pmatrix}, \quad u_3 \leftarrow \begin{pmatrix} -1\\2\\2\\1 \end{pmatrix}$$

$k=2$:

$$u_2 \leftarrow E_2 u_2 = u_2 - (u_1^T u_2)\, u_1 = \begin{pmatrix} 2\\-1\\-1\\1 \end{pmatrix} - \frac{1}{2}(-1)\cdot\frac{1}{2}\begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\\-1\\-1\\1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8+1\\-4+1\\-4+1\\4-1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 9\\-3\\-3\\3 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 3\\-1\\-1\\1 \end{pmatrix}$$

$$u_3 \leftarrow E_2 u_3 = u_3 - (u_1^T u_3)\, u_1 = \begin{pmatrix} -1\\2\\2\\1 \end{pmatrix} - \frac{1}{2}\cdot 2\cdot\frac{1}{2}\begin{pmatrix} 1\\1\\1\\-1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} -2-1\\4-1\\4-1\\2+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3\\3\\3\\3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} -1\\1\\1\\1 \end{pmatrix}$$

$$\|u_2\|^2 = \frac{9}{16}(9+1+1+1) = \frac{9\cdot 12}{16} = \frac{9\cdot 3}{4} \implies \|u_2\| = \frac{3\sqrt{3}}{2}$$

$$u_2 \leftarrow \frac{u_2}{\|u_2\|} = \frac{2}{3\sqrt{3}} \cdot \frac{3}{4}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}$$

Za sad

$$u_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}, \quad u_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}, \quad u_3 = \frac{3}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}$$

$k=3$:

$$u_3 \leftarrow E_3 u_3 = u_3 - (u_2^T u_3)\,u_2 = \frac{3}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix} - \frac{1}{2\sqrt{3}} \cdot \frac{3}{\cancel{2}} \cdot \overset{-1}{\cancel{\overset{\cancel{-2}}{(-4)}}} \cdot \frac{1}{\cancel{2}\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}$$

$$= \frac{3}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix} + \frac{\cancel{3}}{2\cdot\cancel{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-3+3\\3-1\\3-1\\3+1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\2\\2\\4\end{pmatrix} = \begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

$$\|u_3\| = \sqrt{1+1+4} = \sqrt{6}$$

$$u_3 \leftarrow \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$$

Prema tome

$$u_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}, \quad u_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}3\\-1\\-1\\1\end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}.$$

(#) Koristeći Gram-Schmidt-ovu proceduru pronaći ortonormirane baze za četri fundamentalna podprostora od $A=\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -4 & 6 & -2 \\ 3 & -6 & 9 & -3 \end{pmatrix}$.

Rj. Prisjetimo se

im(A) = prostor generisan pomoću kolona matrice A

im(A^T) = prostor generisan pomoću redova matrice A

$$\mathrm{im}(A) := \{Ax \mid x \in \mathbb{R}^n\}$$
$$\mathrm{im}(A) := \{Ay \mid y \in \mathbb{R}^m\} \quad \text{za } A_{m\times n}$$

$$\ker(A) := \{x \mid Ax = \mathbf{0}\}$$

ker(A) = prostor generisan linearno nezavisnim skupom koji čine rješenja linearne jednačine $Ax=0$

ker(A^T) = prostor generisan pomoću zadnjih m-r redova matrice P, gdje je P nesingularna matrica takva da $PA=U$, U u red ešelon obliku, rang(A)=r.

Prvo svedimo A na reducirani red ešelon oblik da bi odredili "obične" baze za svaki od prostora.

$$A=\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -4 & 6 & -2 \\ 3 & -6 & 9 & -3 \end{pmatrix} \underset{III_v + I_v\cdot(-3)}{\overset{II_v + I_v\cdot(-2)}{\sim}} \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_A$$

Nenula redovi od E_A generišu im(A^T)

Osnovne kolone od A generišu im(A)

$$im(A) = span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}, \qquad im(A^T) = \left\{\begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}\right\}.$$

Da bi odredili generator skup za im(A) rješimo sustav

$Ax = 0$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 2 & -4 & 6 & -2 & 0 \\ 3 & -6 & 9 & -3 & 0 \end{array}\right) \underset{III_v + I_v\cdot(-3)}{\overset{II_v + I_v\cdot(-2)}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{rang } A = \text{rang } \bar{A} = 1 < 4$$

3 promjenjive uzimamo proizvoljno

$$x_2 = s, \quad x_3 = t, \quad x_4 = u$$

$$x_1 = 2s - 3t + 4u$$

Rješenje je oblika $$x = \begin{pmatrix} 2s-3t+4u \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix}2\\1\\0\\0\end{pmatrix}s + \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}t + \begin{pmatrix}4\\0\\0\\1\end{pmatrix}u$$

$$ker(A) = span\left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\\1\end{pmatrix}\right\}$$

Sad odredimo matricu P takvu da $PA = U$ tj. svedimo matricu $(A \mid I)$ na oblik $(U \mid P)$.

$$\left(\begin{array}{cccc|ccc} 1 & -2 & 3 & -1 & 1 & 0 & 0 \\ 2 & -4 & 6 & -2 & 0 & 1 & 0 \\ 3 & -6 & 9 & -3 & 0 & 0 & 1 \end{array}\right) \underset{III_v + I_v\cdot(-3)}{\overset{II_v + I_v\cdot(-2)}{\sim}} \left(\begin{array}{cccc|ccc} 1 & -2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 1 \end{array}\right)$$

$$\text{rang } A = 1, \quad A_{3\times 4}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \Rightarrow ker(A^T) = \left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$

Prema tome

$$ker(A) = span\left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\\1\end{pmatrix}\right\}, \quad ker(A^T) = \left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$

Sad primjenimo Gram-Schmidtovu proceduru na svaki od ovih prostora.

a) $\operatorname{im}(A) = \operatorname{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$, $u_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\|u_1\| = \sqrt{1+4+9} = \sqrt{14}$

$$\operatorname{im}(A) = \operatorname{span}\left\{\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$

b) $\operatorname{im}(A^T) = \operatorname{span}\left\{\begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}\right\}$, $u_1 = \begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}$, $\|u_1\|^2 = \sqrt{1+4+9+1} = \sqrt{15}$

$$\operatorname{im}(A^T) = \operatorname{span}\left\{\frac{1}{\sqrt{15}}\begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}\right\}$$

c) $\ker(A) = \left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$, $x_1 = \begin{pmatrix}2\\1\\0\\0\end{pmatrix}$, $x_2 = \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}$, $x_3 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

Klasični Gram-Schmidtov algoritam za $\mathbb{R}^n$

Za $k=1$: $\quad u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $\quad u_k \leftarrow x_k - \displaystyle\sum_{i=1}^{k-1} (u_i^T x_k) u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

$a \leftarrow b$ tumačiti kao a definiši da bude b

$k=1$:

$\|x_1\| = \sqrt{5}$, $\quad u_1 \leftarrow \dfrac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\\0\end{pmatrix}$

$$u_1^T x_2 = \frac{1}{\sqrt{5}}(2\ 1\ 0\ 0)\begin{pmatrix}-3\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\cdot(-6)$$

$-6+0+0+0$

$k=2$:

$u_2 \leftarrow x_2 - (u_1^T x_2) u_1$

$u_2 \leftarrow \dfrac{u_2}{\|u_2\|}$

$$(u_1^T x_2) u_1 = -\frac{6}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\\0\end{pmatrix} = \frac{-6}{5}\begin{pmatrix}2\\1\\0\\0\end{pmatrix}$$

$$u_2 \leftarrow \begin{pmatrix}-3\\0\\1\\0\end{pmatrix} + \frac{6}{5}\begin{pmatrix}2\\1\\0\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-15+12\\0+6\\5+0\\0+0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-3\\6\\5\\0\end{pmatrix}$$

$$\|u_2\| = \frac{1}{5}\cdot\sqrt{9+36+25} = \frac{\sqrt{70}}{5}$$

$$u_2 \leftarrow \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

Prema tome $u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$k=3$:

$$u_3 \leftarrow x_k - (u_1^T x_3) u_1 - (u_2^T x_3) u_2$$

$$u_3 \leftarrow \frac{u_3}{\|u_3\|}$$

$$u_1^T x_3 = \frac{1}{\sqrt{5}} \cdot (2+0+0+0) = \frac{2}{\sqrt{5}}$$

$$(u_1^T x_3) u_1 = \frac{2}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_2^T x_3 = \frac{1}{\sqrt{70}} (-3+0+0+0) = \frac{-3}{\sqrt{70}}$$

$$(u_2^T x_3) u_2 = \frac{-3}{70} \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix}$$

$$x_k - (u_1^T x_3) u_1 - (u_2^T x_3) u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{3}{70} \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{70} \left(70 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 2 \cdot 14 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix} \right) = \frac{1}{70} \begin{pmatrix} 5 \\ -10 \\ 15 \\ 70 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 14 \end{pmatrix}$$

$$u_3 \leftarrow \frac{1}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 14 \end{pmatrix}, \quad \|u_3\| = \frac{1}{14} \sqrt{1+4+9+196} = \frac{\sqrt{210}}{14}$$

$$u_3 = \frac{1}{\sqrt{210}} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 14 \end{pmatrix}.$$

Prema tome $\ker(A) = \operatorname{span} \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ 6 \\ 5 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{210}} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 14 \end{pmatrix} \right\}$

d) ... ZA VJEŽBU ...

Rj:

$$\ker(A^T) = \operatorname{span} \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix} \right\}.$$

(#) Neka je $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{pmatrix}$ i $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Odrediti pravokutnu QR faktorizaciju od A.

(b) Koristeći QR faktorizaciju iz dijela (a) odrediti rješenje za aproksimaciju pomoću najmanjih kvadrata.

Rj.

Posmatrajmo skup $\left\{ x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ 1 \\ -3 \\ 1 \end{pmatrix} \right\}$

Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} (u_i^T x_k) u_i$

$u_k \leftarrow \dfrac{u_k}{\|u_k\|}$

U našem slučaju

$k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

$k=2$: $u_2 \leftarrow x_2 - (u_1^T x_2) u_1$

$u_2 \leftarrow \dfrac{u_2}{\|u_2\|}$

$k=3$: $u_3 \leftarrow x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2$

$u_3 \leftarrow \dfrac{u_3}{\|u_3\|}$

$k=1$: $\quad x_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$, $\quad \|x_1\| = \sqrt{x_1^T x_1} = \sqrt{3}$, $\quad u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$

$k=2$: $\quad u_2 \leftarrow x_2 - (u_1^T x_2)\, u_1 \qquad x_2 = \begin{pmatrix}0\\2\\1\\1\end{pmatrix}$

$u_2 \leftarrow \frac{u_2}{\|u_2\|}$

$$u_1^T x_2 = \frac{1}{\sqrt{3}}(1\ 1\ 1\ 0)\begin{pmatrix}0\\2\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}3 = \frac{3}{\sqrt{3}}$$

$$(u_1^T x_2)\, u_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$u_2 \leftarrow \begin{pmatrix}0\\2\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}, \qquad \|u_2\| = \sqrt{3}$$

$$u_2 \leftarrow \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

Prema tome $\quad u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$, $\quad u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$, $\quad x_3 = \begin{pmatrix}-1\\1\\-3\\1\end{pmatrix}$

$k=3$: $\quad u_3 \leftarrow x_3 - (u_1^T x_3)\, u_1 - (u_2^T x_3)\, u_2$

$u_3 \leftarrow \frac{u_3}{\|u_3\|}$

$$u_1^T x_3 = \frac{1}{\sqrt{3}}(-1+1-3+0) = -\frac{3}{\sqrt{3}}, \qquad (u_1^T x_3)\, u_1 = \begin{pmatrix}-1\\-1\\-1\\0\end{pmatrix}$$

$$u_2^T x_3 = \frac{1}{\sqrt{3}}(-1\ 1\ 0\ 1)\begin{pmatrix}-1\\1\\-3\\1\end{pmatrix} = \frac{1}{\sqrt{3}}(1+1+0+1) = \frac{3}{\sqrt{3}}, \quad (u_2^T x_3)\, u_2 = \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

$$u_3 \leftarrow \begin{pmatrix}-1\\1\\-3\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\-2\\0\end{pmatrix}, \qquad \|u_3\| = \sqrt{6}.$$

Prema tome

$$u_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\quad u_2=\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix},\quad u_3=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}.$$

a) QR Faktorizacija

Svaku matrica $A_{m\times n}$ sa linearno nezavisnim kolonama se može jedinstveno faktorisati kao $A=QR$ gdje su kolone od Q ortonormirana baza za $\operatorname{im}(A)$, a R je gornje trougaona matrica sa pozitivnim dijagonalnim elementima. Kolone od $Q=\begin{pmatrix}| & | & & | \\ q_1 & q_2 & \dots & q_n \\ | & | & & |\end{pmatrix}$ su rezultat primjene Gram-Schmidt-ove procedure na kolone od A $A=\begin{pmatrix}| & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & |\end{pmatrix}$ a R je dat sa

$$R=\begin{pmatrix}\nu_1 & q_1^* a_2 & q_1^* a_3 & \dots & q_1^* a_n \\ 0 & \nu_2 & q_2^* a_3 & \dots & q_2^* a_n \\ 0 & 0 & \nu_3 & \dots & q_3^* a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \nu_n\end{pmatrix}$$

gdje $\nu_1=\|a_1\|, \dots, \nu_k=\|a_k-\sum_{i=1}^{k-1}\langle q_i, a_k\rangle q_i\|$ za $k>1$

U našem slučaju $Q=\begin{pmatrix}1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\\ 0 & 1/\sqrt{3} & 0\end{pmatrix}$ i

$$R=\begin{pmatrix}\sqrt{3} & q_1^T x_2 & q_1^T x_3\\ 0 & \sqrt{3} & q_2^T x_3\\ 0 & 0 & \sqrt{6}\end{pmatrix}.\quad q_1^T x_2=\frac{1}{\sqrt{3}}(1\ 1\ 1\ 0)\begin{pmatrix}0\\2\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}\cdot 3$$

$$q_1^T x_3 = \frac{1}{\sqrt{3}}(1\ 1\ 1\ 0)\begin{pmatrix}-1\\1\\-3\\1\end{pmatrix} = -\frac{3}{\sqrt{3}}$$

$$q_2^T x_3 = \frac{1}{\sqrt{3}}(-1\ 1\ 0\ 1)\begin{pmatrix}-1\\1\\-3\\1\end{pmatrix} = \frac{3}{\sqrt{3}}$$

$$R = \begin{pmatrix}\sqrt{3} & \sqrt{3} & -\sqrt{3}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{6}\end{pmatrix}$$

b) Aproksimacija pomoću najmanjih kvadrata

Neka je $A \in Mat_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$ i neka je $\epsilon = \epsilon(x) = Ax - b$. Problem kako pronaći vektor x koji će minimizirati vrijednost

$$\sum_{i=1}^{m} \epsilon_i = \epsilon^T \epsilon = (Ax-b)^T(Ax-b)$$

zovemo aproksimacija pomoću najmanjih kvadrata.

Linearni sistemi i QR faktorizacija

Ako je $rang(A_{m\times n}) = n$, i ako je $A = QR$ QR faktorizacija, tada rješenje od nesingularnog trougaonog sistema $Rx = Q^T b$ je ili rješenje od $Ax = b$ ili rješenje za aproksimaciju pomoću najmanjih kvadrata od $Ax = b$, u zavisnosti da li je $Ax = b$ saglasan sistem.

Sistem $QRx = Q^T b$ (gornje trougaoni sistem) nije teško riješiti (ZA VJEŽBU)

$$x = \begin{pmatrix}2/3\\1/3\\0\end{pmatrix}$$

(#) Primjeniti Gram-Schmidt-ovu proceduru sa standardnim unutrašnjim proizvodom za $\mathbb{C}^3$ na $\left\{ \begin{pmatrix} i \\ i \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \right\}$.

Rj.: <u>Klasični Gram-Schmidtov algoritam</u>

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} (u_i^* x_k) u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

U našem slučaju $x_1 = \begin{pmatrix} i \\ i \\ i \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$ pa je

$k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x\|_1}$

$k=2$: $u_2 \leftarrow x_2 - (u_1^* x_2) u_1$

$u_2 \leftarrow \dfrac{u_2}{\|u_2\|}$

$k=3$: $u_3 \leftarrow x_3 - (u_1^* x_3) u_1 - (u_2^* x_3) u_2$

$u_3 \leftarrow \dfrac{u_3}{\|u_3\|}$

$x_1 = \begin{pmatrix} i \\ i \\ i \end{pmatrix}$, $\|x_1\| = \sqrt{x_1^* x_1} = \sqrt{(-i \; -i \; -i)\begin{pmatrix} i \\ i \\ i \end{pmatrix}} = \sqrt{-i^2 - i^2 - i^2} = \sqrt{3}$

$u_1 \leftarrow \dfrac{1}{\sqrt{3}} \begin{pmatrix} i \\ i \\ i \end{pmatrix}$

$u_1^* x_2 = \dfrac{1}{\sqrt{3}} (-i \; -i \; -i) \begin{pmatrix} 0 \\ i \\ i \end{pmatrix} = \dfrac{2}{\sqrt{3}}$

$u_2 \leftarrow x_2 - (u_1^* x_2) u_1$

$(u_1^* x_2) u_1 = \dfrac{2}{3} \begin{pmatrix} i \\ i \\ i \end{pmatrix}$

$$u_2 \leftarrow \begin{pmatrix} 0 \\ i \\ i \end{pmatrix} - \frac{2}{3}\begin{pmatrix} i \\ i \\ i \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -2i \\ 3i & -2i \\ 3i & -2i \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix}, \quad \|u_2\| = \sqrt{\frac{1}{9}\cdot 6} = \frac{\sqrt{6}}{3}$$

$$u_2 \leftarrow \frac{\frac{1}{3}}{\frac{\sqrt{6}}{3}}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix}$$

$$u_3 \leftarrow x_3 - (u_1^* x_3)u_1 - (u_2^* x_3)u_2$$

$$u_1^* x_3 = \frac{1}{\sqrt{3}}(-i \;\; -i \;\; -i)\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} = \frac{1}{\sqrt{3}}, \quad (u_1^* x_3)u_1 = \frac{1}{3}\begin{pmatrix} i \\ i \\ i \end{pmatrix}$$

$$u_2^* x_3 = \frac{1}{\sqrt{6}}(2i \;\; -i \;\; -i)\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} = \frac{1}{\sqrt{6}}, \quad (u_2^* x_3)u_2 = \frac{1}{6}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix}$$

$$u_3 \leftarrow \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} - \frac{1}{3}\begin{pmatrix} i \\ i \\ i \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 - 2i + 2i \\ 0 - 2i - i \\ 6i - 2i - i \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 \\ -3i \\ 3i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -i \\ i \end{pmatrix}$$

$$\|u_3\| = \sqrt{\frac{1}{4}\cdot(1+1)} = \frac{\sqrt{2}}{2} \qquad u_3 \leftarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ i \end{pmatrix}$$

Prema tome

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} i \\ i \\ i \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -2i \\ i \\ i \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ i \end{pmatrix}.$$

(#) Objasniti šta će se desiti kada se Gram-Schmidtov proces primjeni na ortonormirani skup vektora.

Rj.:

Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \frac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$u_k \leftarrow \frac{u_k}{\|u_k\|}$

Pa neka je $\mathcal{B}=\{x_1, x_2, \dots, x_n\}$ ortonormiran skup vektora. Primjenimo Gram-Schmidtov algoritam

$k=1$: $\|x_1\|=1 \Rightarrow u_1 \leftarrow x_1 \Rightarrow u_1 = x_1$

$k=2$:

$$u_2 \leftarrow x_2 - \underbrace{\langle u_1, x_2 \rangle}_{=\langle x_1, x_2 \rangle = 0} u_1 \Rightarrow u_2 \leftarrow x_2$$

$$\|u_2\| = \|x_2\| = 1 \Rightarrow u_2 = x_2$$

$\vdots$

$k=n$: $$u_n \leftarrow x_n - \underbrace{\langle u_1, x_n \rangle}_{=0} u_1 - \underbrace{\langle u_2, x_n \rangle}_{=0} u_2 - \dots - \underbrace{\langle u_{n-1}, x_n \rangle}_{=0} u_{n-1}$$

$$u_n \leftarrow x_n, \quad \|u_n\| = \|x_n\| = 1. \Rightarrow u_n = x_n.$$

Prema tome ako Gram-Schmidtov proces primjenimo na ortonormiran skup vektora neće se desiti ništa. Ortonormirani skup vektora koji se dobije kao rezultat je isti kao i orginalni.

(#) Objasniti šta će se desiti kada se Gram-Schmidtov proces primjeni na linearno zavisan skup vektora.

Rj.

Gram-Schmidt-ov proces ortogonalizacije

Ako je $\mathcal{B}=\{x_1, x_2, \dots, x_n\}$ baza za unitarni prostor $\mathcal{S}$, tada Gram-Schmidt-ov niz definisan sa

$$u_1 = \frac{x_1}{\|x_1\|} \quad i \quad u_k = \frac{x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i}{\left\| x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i \right\|} \quad \text{za } k=2,\dots,n$$

je ortonormirana baza za $\mathcal{S}$.

Algoritam će pasti na prvom vektoru za koji vrijedi

$$x_k \in span\{x_1, x_2, \dots, x_{k-1}\}$$

zato što, ako $x_k \in span\{x_1, x_2, \dots, x_{k-1}\} = span\{u_1, u_2, \dots, u_{k-1}\}$ tada će Furijer-ov razvoj od x_k u odnosu na $span\{u_1, u_2, \dots, u_{k-1}\}$ biti $x_k = \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

pa, prema tome

$$u_k = \frac{x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i}{\left\| x_k - \sum \langle u_i, x_k \rangle u_i \right\|} = \frac{0}{\|0\|}$$

nije definisano.

(a) Primjeniti klasični Gram-Schmidtov algoritam, koristeći aritmetiku sa tri decimalna mjesta, na

$$\left\{ x_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-2} \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} \right\}.$$

Možete pretpostaviti da je $\sqrt{2} \approx 1,41$.

(b) Ponovo koristeći aritmetiku sa tri decimalna mjesta, primjeniti modificirani Gram-Schmidt-ov algoritam na $\{x_1, x_2, x_3\}$, i porediti rezultate sa dijelom (a).

Rj.

a) Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \displaystyle\sum_{i=1}^{k-1} (u_i^T x_k) u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

U našem slučaju

$k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

$k=2$: $u_2 \leftarrow x_2 - (u_1^T x_2) u_1, \quad u_2 \leftarrow \dfrac{u_2}{\|u_2\|}$

$k=3$: $u_3 \leftarrow x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2, \quad u_3 \leftarrow \dfrac{u_3}{\|u_3\|}$

$k=1$:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}, \quad \|x_1\| = \sqrt{1+0+10^{-6}} \approx 1 \quad \Rightarrow \quad u_1 \leftarrow x_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}$$

$k=2$:

$$u_1^T x_2 = (1 \ 0 \ 10^{-3}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1,$$

$$u_2 \leftarrow x_2 - (u_1^T x_2) u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -10^{-3} \end{pmatrix}, \quad \|u_2\| = \sqrt{10^{6}} = 10^{3}$$

$$u_2 \leftarrow \frac{u_2}{\|u_2\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$k=3$:

$$u_1^T x_3 = \begin{pmatrix} 1 & 0 & 10^{-3} \end{pmatrix} \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} = 1, \quad u_2^T x_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} = 0$$

$$(u_1^T x_3) u_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}$$

$$u_3 \leftarrow x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-3} \\ -10^{-3} \end{pmatrix}$$

$$\|u_3\| = \sqrt{0 + 10^{-6} + 10^{-6}} = \sqrt{2 \cdot 10^{-6}} = 10^{-3} \sqrt{2} \approx 10^{-3} \cdot 1{,}41$$

$$u_3 \leftarrow \frac{1}{10^{-3} \cdot 1{,}41} \cdot \begin{pmatrix} 0 \\ 10^{-3} \\ -10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0{,}709 \\ -0{,}709 \end{pmatrix}$$

Prema tome, rezultat klasičnog Gram-Schmidtovog algoritma, koristeći aritmetiku sa tri decimalna mjesta, je

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0.709 \\ -0.709 \end{pmatrix}$$

što nije baš dobro, zato što u_2 i u_3 nisu čak ni približno ortogonalni.

(b) Modificirani Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \frac{x_1}{\|x_1\|}$, $u_j \leftarrow x_j$ za $j = 2, 3, \dots, n$

Za $k > 1$ $\quad u_j \leftarrow u_j - (u_{k-1}^* u_j) u_{k-1}$ za $j = k, k+1, \dots, n$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

U našem slučaju:

$k=1$: $\quad u_1 \leftarrow \frac{x_1}{\|x_1\|}, \quad u_2 \leftarrow x_2, \quad u_3 \leftarrow x_3$

$k=2$:
$$u_2 \leftarrow u_2 - (u_1^* u_2)\, u_1$$
$$u_3 \leftarrow u_3 - (u_1^* u_3)\, u_1$$
$$u_2 \leftarrow \frac{u_2}{\|u_2\|}$$

$k=3$:
$$u_3 \leftarrow u_3 - (u_2^* u_3)\, u_2$$
$$u_3 \leftarrow \frac{u_3}{\|u_3\|}$$

$$\|x_1\| = \sqrt{1+0+10^{-6}} \approx 1$$

$k=1$:
$$u_1 \leftarrow \frac{x_1}{\|x_1\|} = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}, \quad u_2 \leftarrow x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 \leftarrow x_3 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix}$$

$k=2$:
$$u_1^T u_2 = \begin{pmatrix} 1 & 0 & 10^{-3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1, \quad u_1^T u_3 = \begin{pmatrix} 1 & 0 & 10^{-3} \end{pmatrix} \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} = 1$$
$$u_2 \leftarrow u_2 - (u_1^T u_2) u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -10^{-3} \end{pmatrix}, \quad \|u_2\| = 10^{-3}$$
$$u_3 \leftarrow \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-3} \\ -10^{-3} \end{pmatrix}, \quad u_2 \leftarrow \frac{1}{10^{-3}} \begin{pmatrix} 0 \\ 0 \\ -10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$k=3$:
$$u_2^T u_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 10^{-3} \\ -10^{-3} \end{pmatrix} = 10^{-3}$$
$$u_3 \leftarrow \begin{pmatrix} 0 \\ 10^{-3} \\ -10^{-3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -10^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-3} \\ 0 \end{pmatrix}, \quad \|u_3\| = 10^{-3}$$

Prema tome, modificirani Gram-Schmidtov algoritam daje

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

što je dovoljno blizu ortonormiranom skupu s obzirom da smo koristili aritmetiku sa 3 decimalna mjesta.

Zadaci za vježbu

1. Za dati linearno nezavisan skup vektora $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ u unitarnom prostoru, neka je $\mathcal{S}_k = \mathrm{span}\{x_1, x_2, \ldots, x_k\}$ za $k = 1, 2, \ldots, n$. Matematičkom indukcijom pokazati da ako je $\mathcal{O}_k = \{u_1, u_2, \ldots, u_k\}$ Gram-Schmidtov niz (definisan ranije), tada je $\mathcal{O}_k$ zaista ortonormirana baza za $\mathcal{S}_k = \mathrm{span}\{x_1, x_2, \ldots, x_k\}$ za svaki $k = 1, 2, \ldots, n$.

2. Dokazati da ako je $\mathrm{rang}(A_{m\times n}) = n$, tada je pravougaona QR faktorizacija od A jedinstvena, Tj. ako je $A = QR$, gdje $Q_{m\times n}$ ima ortonormirane kolone i $R_{n\times n}$ je gornje trougaona sa pozitivnim dijagonalnim elementima tada su Q i R jedinstvene.

3. Neka je $\mathcal{V}$ unitarni prostor realno-vrijednosnih neprekidnih f-ja definisanih na intervalu $[-1, 1]$, gdje je unutrašnji proizvod definisan sa

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

i neka je $\mathcal{S}$ podprostor od $\mathcal{V}$ koji je generisan sa tri linearno nezavisna polinoma $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = x$, $\mathcal{L}_2 = x^2$.

(a) Koristeći Gram-Schmidtov proces odrediti ortonormiran skup polinoma $\{P_0, P_1, P_2\}$ koji generišu $\mathcal{S}$. Dobijeni polinomi su prva tri normirana Legendre-ova polinoma.

(b) Provjeriti da li P_n-ovi zadovoljavaju Legendre-ovu diferencijalnu jednačinu $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ za $n = 0, 1, 2$. Ova jednačina i njezina rješenja su od velike važnosti u primjenjenoj matematici.